Signed Opinion Dynamics on Networks: Opposing vs. Repelling, Information, and Balance

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MMSS Thesis (Advisor: Ben Golub)

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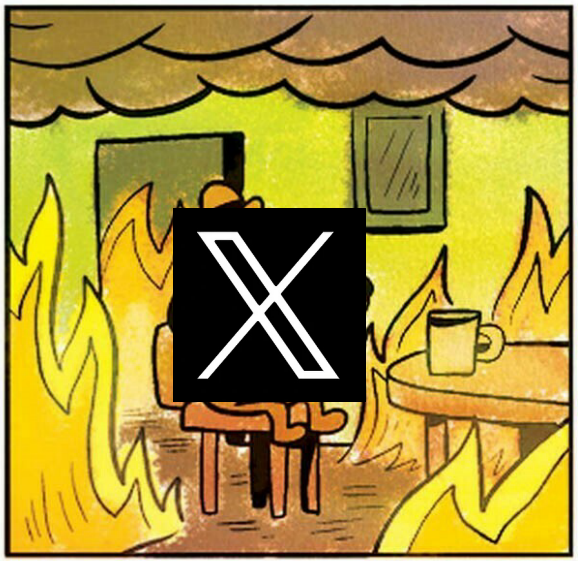
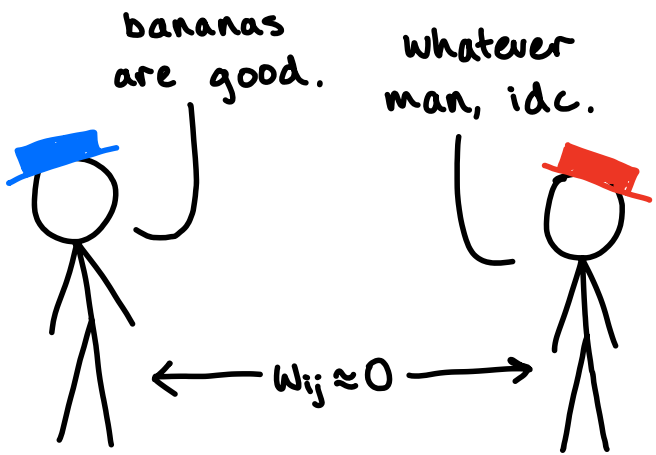
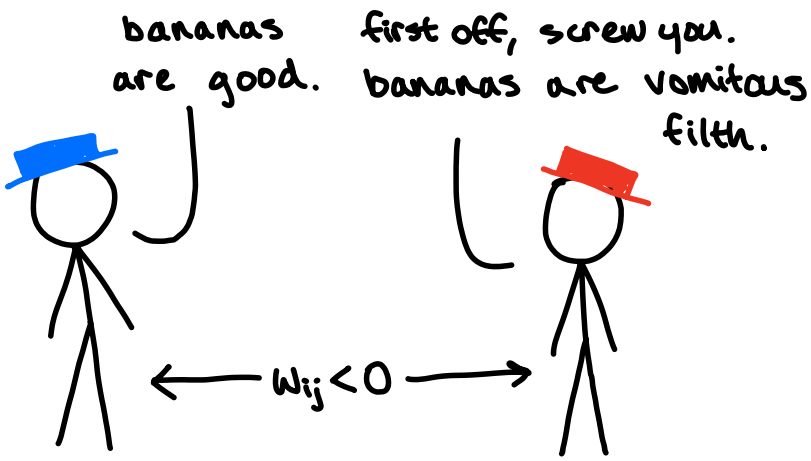
People Don't Like Their Opps



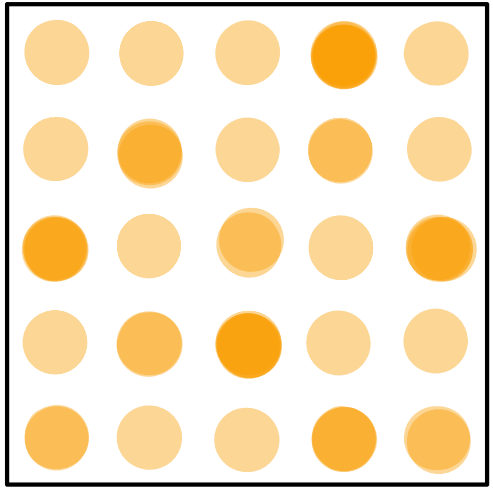




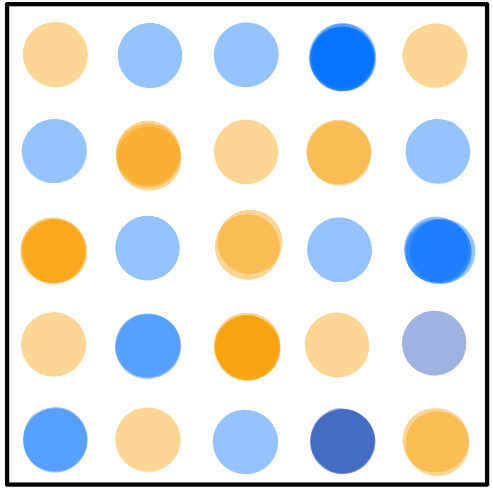
Background: Antagonism

- In "DeGroot-type" models, you only have friends:
  - Update rule:
$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t)$$
  - Matrix form:
$$\mu(t+1) = W\mu(t)$$
    (where W row-stochastic, nonnegative)
- In antagonistic models, you also have opps:
  - Opposing rule:
$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} (-1)w_{ij}\mu_j(t)$$
  - Repelling rule:
$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} w_{ij}(\mu_i(t) - \mu_j(t))$$
  - Matrix form:
$$\mu(t+1) = \tilde{W}\mu(t)$$

Background: Antagonism



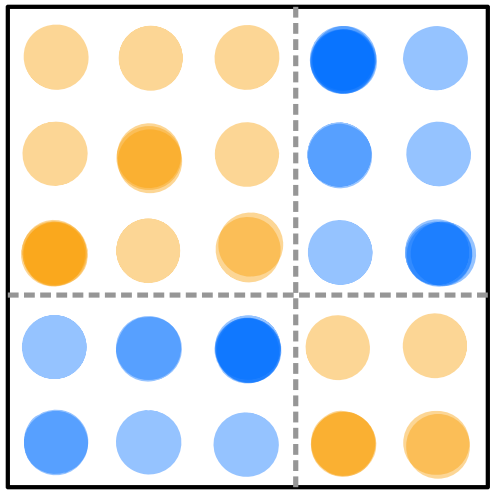
$W \geq 0$



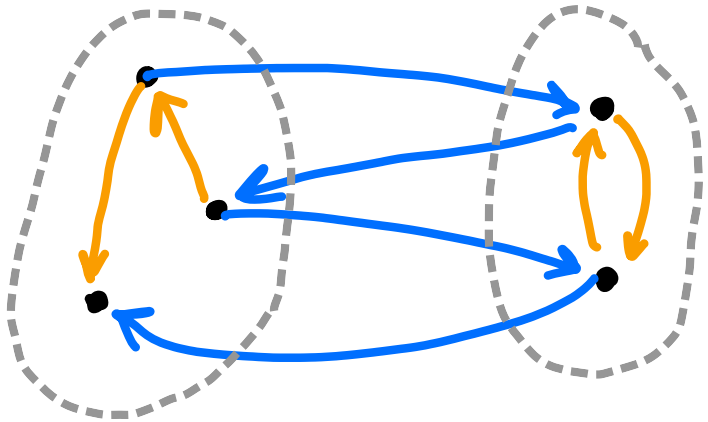
$\tilde{W}$ signed

Background: Structural Balance (SB)

- Intuitively: "enemy of my enemy is my friend"
- Formally: weight product of every cycle is nonnegative



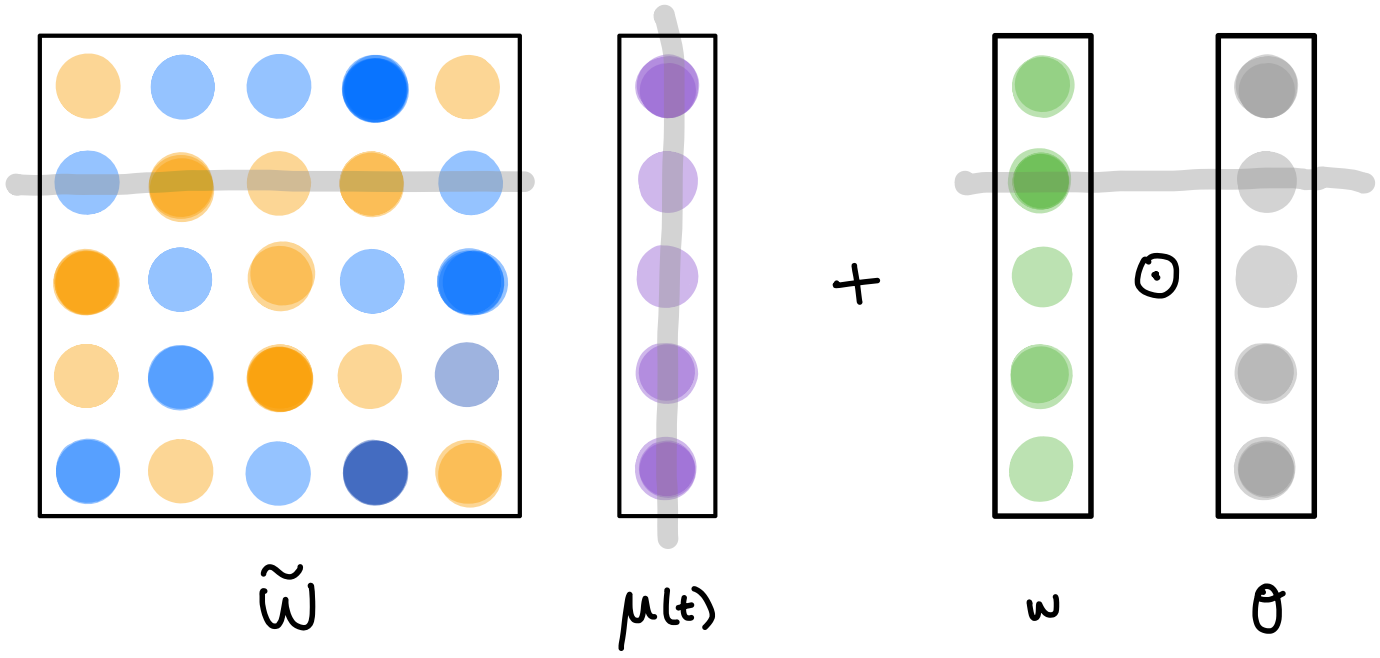
$\tilde{W}$ is SB



Background: Information

- Idea: your opinion is also influenced by external sources of info (eg. media), not just your neighbors
- Formally:

$$\mu(t) = \tilde{W}\mu(t) + w \odot \theta(t)$$

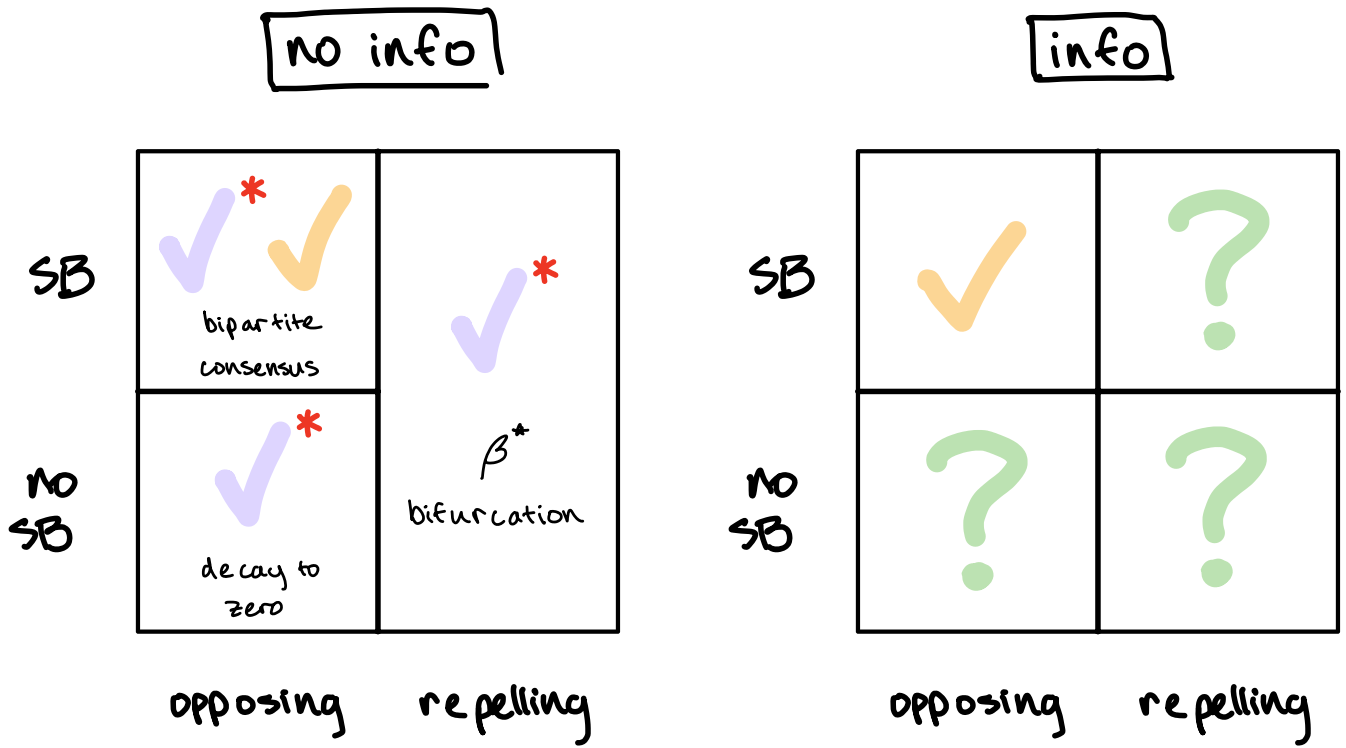


Existing Literature

- **Shi et. al (2019)**
  - Foundational discrete-time analysis, β^* threshold
  - Scope: No info regime
  - Caveats: only have sharp threshold for symmetric $\tilde{W}$; formulate $\tilde{W}$ as a "weight-uniform" (I – weighted Laplacian)
- **Fontan and Altafini (2021)**
  - Continuous time Lyapunov analysis, threshold in "social effort" parameter
- **Lena, Merlino, Zenou ("LMZ") (2025)**
  - Introduce information
  - Scope: Opposing, structurally balanced
  - Caveats: scope is limited, constant info
- **DeMarzo, Vayanos, Zwiebel ("DVZ") (2003)**
  - Persuasian bias (nonnegative $\tilde{W}$), unidimensional opinions

Existing Literature: Goal

- **Shi et. al (2019)**
- **Lena, Merlino, Zenou ("LMZ") (2025)**



Methodology

1. **Understand "Kernel of Niceness"**
   - What conditions on the update matrix/rule (eg. (absolute) (sub) row-stochasticity, c-summing, etc) are sufficient and/or necessary for "nice" long-run behavior (eg. single-value consensus, bipartite consensus, bounded opinions)[1]?
2. **Design Generalized Model**
   - Parsimonious (general), sociologically interpretable conditions
   - Infinite dimensional vector spaces? Bayesian updating interpretation?
   - If model turns out to be "bad" (not just "generating unfortunate behavior" but pathologically flawed), revisit Step 1
3. **Execute Model**
   - Determine target behaviors (eg. consensus, divergence, periodicity, etc.) and the regimes of parameter space that they induce (eg. $\beta*$ bifurcations)
   - Do this for each "toggling" (SB/no SB, opp./rep.)
4. **Computational Simulations (time permitting)**

[1] These often correspond to spectral/operator norm properties

Progress Thus Far

- Showed that (contrary to LMZ claim), opp + no info does not reach bipartite consensus, but rather damps to 0 vector[2]
- Reduced LMZ (i.e. opp + abs row stochastic + SB) to DeGroot (mostly just for myself)
- Generalized a lemma from Shi (SB iff $||\tilde{W}||_{\text{op}} = ||W||_{\text{op}}$; SB implies $r(\tilde{W}) = r(W)$)
- Derived a sufficient (but not necessary) condition to get from c-summing to spectral radii properties
  - At this point, I could propose a new model subject to this condition that would make the info case under repelling dynamics "'nice", but I think this condition is a bit strong
- Currently trying to understand the case where $r(\tilde{W}) \geq 1$ for repelling dynamics

[2] Technically I only showed this for symmetric $\tilde{W}$, but the core task of showing principle eigenvalue in the open disk is done

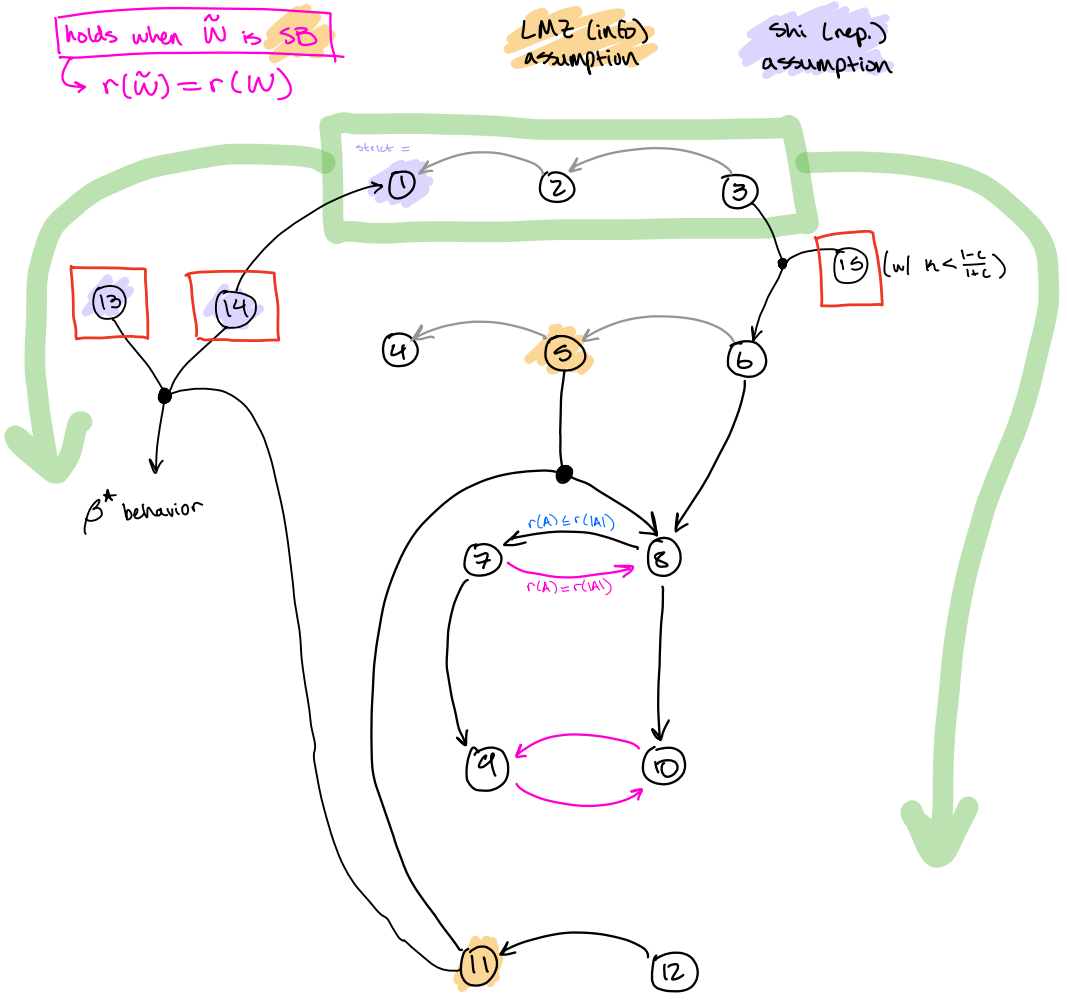
Implication Graph

① $\tilde{W}\mathbb{1} \leq \mathbb{1}$
② $\tilde{W}\mathbb{1} \nleq \mathbb{1}$
③ $\tilde{W}\mathbb{1} \leq c\mathbb{1}$
④ $W\mathbb{1} \leq \mathbb{1}$
⑤ $W\mathbb{1} \nleq \mathbb{1}$
⑥ $W\mathbb{1} \leq c\mathbb{1}$
⑦ $r(\tilde{W}) < 1$
⑧ $r(W) < 1$
⑨ $(I - \tilde{W})$ invertible
⑩ $(I - W)$ invertible
⑪ $\tilde{W}$ irreducible
⑫ $\tilde{W}$ primitive
⑬ $\tilde{W}$ symmetric
⑭ $\tilde{W} = I - \alpha L_{sym} - \beta L_{sym}^2$
⑮ $\exists\, \kappa \in [0,1)$ s.t. $\forall$ rows i,
$\underbrace{\sum_{j:\tilde{W}_{ij}<0} -\tilde{W}_{ij}}_{=:N_i} \leq \kappa \underbrace{\sum_{j:\tilde{W}_{ij}>0} W_{ij}}_{=:P_i}$

holds when $\tilde{W}$ is SD
↳ $r(\tilde{W}) = r(W)$

LMZ (info) assumption

Shi (rep.) assumption



□ too strong

summing properties of $\tilde{W}$

summing properties of W

spectral radii properties

invertibility properties